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The Pseudo-Integral of a System of Differential Equations*

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I. INTRODUCTION

First we shall briefly review what is meant by saying that a function is an integral for a system of differential equations. Next we shall introduce a generalization of this idea by modifying two conditions on the function to obtain what in this paper shall be called the pseudo-integral. Finally we shall apply this new concept to some problems in celestial mechanics to demonstrate its usefulness in obtaining information about the solution of a system of differential equations.

II. INTEGRAL

Let Γ be the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{x}$ belongs to E^n, an n-dimensional Euclidean space. t belongs to T, a one-dimensional Euclidean space. f^i and $\partial f^i/\partial x^j$ for $i, j = 1, 2 \ldots n$ are continuous in a domain D' of $E^n \times T$ containing the point $(\hat{\mathbf{x}}, \hat{t})$, and the dot represents a derivative with respect to t.

From the Theory of Differential Equations there exists domains S in E^n and I in T containing $\hat{\mathbf{x}}$ and $\hat{t}$ respectively, and a function $\mathbf{x}(\mathbf{w}, t)$ defined in $S \times I$ such that

(i) The set of points $\{(\mathbf{x}(\mathbf{w}, t), t)\}$ for all $(\mathbf{w}, t)$ in $S \times I$ is a domain D contained in D'.

(ii) $\mathbf{x} = \mathbf{x}(\mathbf{w}, t)$ is the unique solution to Γ in I for which $\mathbf{w} = \mathbf{x}(\mathbf{w}, \hat{t})$.

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(iii) $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$ the inverse of $\mathbf{x} = \mathbf{x}(\mathbf{w}, t)$ is defined and continuous for all points of D.

The function $F = F(\mathbf{x}, t)$ is called an integral of Γ if (i) $F(\mathbf{x}, t)$ has continuous first partial derivatives in D, and (ii) $F(\mathbf{x}(\mathbf{w}, t), t) = F(\mathbf{w}, \hat{t})$ identically in $S \times$ I. Assuming (i), (ii) is equivalent to the following condition:

$$\frac{dF}{dt} = \frac{\partial F}{\partial x^i}(\mathbf{x}(\mathbf{w}, t), t) f^i(\mathbf{x}(\mathbf{w}, t), t) + \frac{\partial F}{\partial t}(\mathbf{x}(\mathbf{w}, t), t) = 0$$

identically in $S \times I$ where $\partial F/\partial t$ and dF/dt are the derivatives of F with respect to t keeping $\mathbf{x}$ and $\mathbf{w}$ fixed respectively.

Substituting $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$ into this expression we obtain

$$\frac{\partial F}{\partial x^i}(\mathbf{x}, t) f^i(\mathbf{x}, t) + \frac{\partial F}{\partial t}(\mathbf{x}, t) = 0$$

identically in D. Therefore, if is not necessary to know the general solution of Γ in order to determine whether a given function is an integral. This coupled with the fact that for the particular solution $\mathbf{x} = \mathbf{x}(\mathbf{w}_0, t)$, where $\mathbf{w}_0$ belongs to S,

$$F(\mathbf{x}, t) = F(\mathbf{w}_0, \hat{t})$$

makes this concept very useful.

III. The Pseudo-Integral

Let us consider the function $F(\mathbf{x}, \mathbf{w}, t)$ such that (i) for any value of $\mathbf{w}$ in S, $\partial F/\partial x^i$ and $\partial F/\partial t$ are continuous in D and (ii) $F(\mathbf{x}(\mathbf{u}, t), \mathbf{u}, t) = F(\mathbf{u}, \mathbf{u}, \hat{t})$ identically for all t in I and all $\mathbf{u}$ in U, where U is the set of all $\mathbf{w}$ in S that satisfy the k (where k may be 0) equations of constraints:

$$h_j(\mathbf{w}) = 0 \qquad \text{for} \qquad j = 1, 2, \cdots k.$$

If $k = 0$ and F does not depend explicitly on $\mathbf{w}$ then F is a pseudo-integral if and only if it is an integral.

For convenience we shall call $G(\mathbf{x}, \mathbf{w}, t)$ a normalized pseudo-integral if it is a pseudo-integral and if $G(\mathbf{u}, \mathbf{u}, \hat{t}) = 0$ identically for all $\mathbf{u}$ in U. Given a pseudo-integral $F(\mathbf{x}, \mathbf{w}, t)$ we can obtain a normalized pseudo-integral $G(\mathbf{x}, \mathbf{w}, t)$ by defining G as follows:

$$G(\mathbf{x}, \mathbf{w}, t) = F(\mathbf{x}, \mathbf{w}, t) - F(\mathbf{w}, \mathbf{w}, \hat{t}).$$

Before we consider specific examples, let us indicate in general what our procedure shall be. For the purpose of this discussion, we shall assume that for any $\mathbf{w}$ in S all functions of $\mathbf{x}$, $\mathbf{w}$, and t, including $\mathbf{f}(\mathbf{x}, t)$, have sufficiently many derivatives with respect to $\mathbf{x}$ and t that are continuous in D. Given a function $G(\mathbf{x}, \mathbf{w}, t)$ let us define $G_0(\mathbf{x}, \mathbf{w}, t) = G(\mathbf{x}, \mathbf{w}, t)$ and

$$G_l(\mathbf{x}, \mathbf{w}, t) = \frac{\partial G_{l-1}}{\partial x^i} f^i + \frac{\partial G_{l-1}}{\partial t}$$

where l is any integer greater than 0. Thus

$$G_l(\mathbf{x}(\mathbf{w}, t), \mathbf{w}, t) = \frac{d^l G}{dt^l}(\mathbf{x}(\mathbf{w}, t), \mathbf{w}, t)$$

i.e. G_l is the lth derivative of G along the solution curves of Γ. G is a normalized pseudo-integral of Γ only if $G_p(\mathbf{x}(\mathbf{u}, t), \mathbf{u}, t) = 0$ for all $p \geq 0$ and all $(\mathbf{u}, t)$ in $U \times I$. One might expect that the infinite system of equations $G_q(\mathbf{x}, \mathbf{u}, t) = 0$ for all $q \geq 0$ would impose too many conditions on the variables $\mathbf{x}$, $\mathbf{u}$ and t, i.e, would have no solution. However, if there is an integer s such that[1]

$$G_s(\mathbf{x}, \mathbf{u}, t) = \sum_{p=0}^{s-1} a_p(\mathbf{x}, \mathbf{u}, t)\, G_p(\mathbf{x}, \mathbf{u}, t)$$

and a value of $(\mathbf{x}, \mathbf{u}, t)$ for which $G_p(\mathbf{x}, \mathbf{u}, t) = 0$ if $p < s$; then for all q and the same $(\mathbf{x}, \mathbf{u}, t)$, $G_q(\mathbf{x}, \mathbf{u}, t) = 0$. If in addition $G_p(\mathbf{u}, \mathbf{u}, \hat{t}) = 0$ identically for $p < s$ and all $\mathbf{u}$ in U, G is a normalized pseudo-integral since it satisfies the ordinary linear differential equation

$$\frac{d^s G}{dt^s} = \sum_{p=0}^{s-1} a_p(\mathbf{x}(\mathbf{u}, t), \mathbf{u}, t) \frac{d^p G}{dt^p}$$

containing the parameter $\mathbf{u}$ and subject to the initial conditions

$$\frac{d^p G}{dt^p}(\mathbf{u}, \mathbf{u}, \hat{t}) = 0 \qquad \text{for} \qquad p = 0, 1 \cdots s-1.$$

We see that if $G(\mathbf{x}, \mathbf{w}, t)$ is a normalized pseudo-integral so is $G_p(\mathbf{x}, \mathbf{w}, t)$ for any p and the same constraints. In general a set of functions $\{G^\alpha\}$ where α belongs to some set are called simultaneous pseudo-integrals if they are all pseudo-integrals for the same set of constraints. In practice the equations of constraints shall not usually be given a priori. Rather, if there exists a relationship of the form

$$G_s(\mathbf{x}, \mathbf{w}, t) = \sum_{p=0}^{s-1} a_p(\mathbf{x}, \mathbf{w}, t)\, G_p(\mathbf{x}, \mathbf{w}, t)$$

[1] For example if $G_{s-1} = H(G_0, G_1, \cdots, G_{s-2})$, we may take

$$a_p = \frac{\partial H}{\partial G_{p-1}}(G_0(\mathbf{x}, \mathbf{u}, t), G_1(\mathbf{x}, \mathbf{u}, t), \cdots, G_{s-2}(\mathbf{x}, \mathbf{u}, t))$$

for $1 \leq p \leq s-1$ and $a_0 = 0$.

then the equations of constraints can be taken to be $G_p(\mathbf{w}, \mathbf{w}, \hat{t}) = 0$ for $p < s$, if these equations have a solution. Also in practice we may consider a function $G(\mathbf{x}, \mathbf{w}, a_1, a_2, \cdots a_m, t)$ where a_i for $i = 1, 2 \cdots m$ is a function of $\mathbf{w}$ to be so chosen that if possible G is a pseudo-integral. These ideas shall be made more concrete in the following examples.

IV. Examples

Let

$$\ddot{r}_i = -g(m + m_i)\frac{r_i}{|r_i|^3} - g\sum_{\substack{l=1\\ l\neq i}}^{k}{}' m_l\left[\frac{r_i - r_l}{|r_i - r_l|^3} + \frac{r_l}{|r_l|^3}\right]$$

where r_i is a **vector** in a plane with coordinates (x_i, y_i), and m, m_i and g are greater than 0 for $i = 1, 2, \cdots k$. These are the equations of motion of $k + 1$ bodies in a coordinate system with origin on the body of mass m and axes parallel to an inertial system, under the assumptions that the bodies always lie in a plane and the only forces are the gravitational interaction between the bodies. r_i is the position vector of the body of mass m_i.

A. First Example

As our first example we shall consider the case $k = 1$. For this case the equations are

$$\ddot{r} = -g(m + m_1)\frac{r}{|r|^3}$$

where for convenience the subscript on r_1 has been dropped since $k = 1$.

Let

$$G = Ax + By + D|r| - D^2$$

Therefore,

$$\dot{G} = A\dot{x} + B\dot{y} + D\frac{r \cdot \dot{r}}{|r|}$$

$$\ddot{G} = -\frac{g(m + m_1)}{|r|^3}\left[Ax + By + D|r| - \frac{D}{g(m + m_1)}(\dot{r}^2r^2 - (r \cdot \dot{r})^2)\right]$$

$$\ddot{G} = -\frac{g(m + m_1)}{|r|^3}\left[G + D^2 - \frac{D}{g(m + m_1)}(r \times \dot{r})^2\right]$$

$$\dddot{G} = |r|^3\left[\frac{d}{dt}\left(\frac{1}{|r|^3}\right)\right]\ddot{G} - \frac{g(m + m_1)}{|r|^3}\dot{G}$$

Setting $G = \dot{G} = \ddot{G} = 0$ at $t = \hat{t}$ we obtain

$$D^2 = \frac{D}{g(m + m_1)} (r_0 \times \dot{r}_0)^2$$

where we use r_0 and $\dot{r}_0$ with components (x_0, y_0) and $(\dot{x}_0, \dot{y}_0)$ respectively instead of w. Let us set $D = [1/g(m + m_1)] (r_0 \times \dot{r}_0)^2$. Since $\Delta^2 = g(m_1 + m) D$ where Δ is the determinant $\begin{vmatrix} x_0 & y_0 \\ \dot{x}_0 & \dot{y}_0 \end{vmatrix}$, A, B, and D can be be determined in terms of r_0 and $\dot{r}_0$ so that G is a normalized pseudo-integral for all solutions to the two-body problem, i.e., all solutions lie on a straight line or on (the branch of) a conic section. This is of course a well known fact in celestial mechanics although G is not an integral since A, B, and D depend on initial conditions.

It is interesting to note that if the most general Second order polynomial in x and y was chosen as a possible normalized pseudo-integral for this problem instead of the selection made above, the same results could have been obtained in a similar fashion.

B. *Second Example*

As our second example let us see if there is a solution to the planar three body problem such that the triangle formed by the bodies is always isosceles, but not necessarily congruent to its initial configuration, i.e., if $G = r_1^2 - r_2^2$ is a normalized pseudo-integral (in the coordinate system we are using the body of mass m is at the origin) for the equations

$$\ddot{r}_1 = -g(m + m_1) \frac{r_1}{|r_1|^3} - gm_2 \left[\frac{r_1 - r_2}{|r_1 - r_2|^3} + \frac{r_2}{|r_2|^3}\right]$$

$$\ddot{r}_2 = -g(m + m_2) \frac{r_2}{|r_2|^3} - gm_1 \left[\frac{r_2 - r_1}{|r_2 - r_1|^3} + \frac{r_1}{|r_1|^3}\right]$$

In deriving a differential equation for G we may make use of relationships such as

$$\frac{1}{|r_2|} = \frac{1}{|r_1|} + \frac{G}{|r_1|\,|r_2|\,(|r_1| + |r_2|)}$$

However, our work will be simplified if we make use of the fact that G is a normalized pseudo-integral for the given system of equations only if G is a normalized pseudo-integral for the following system:

$$\ddot{r}_1 = -g(m + m_1) \frac{r_1}{|r_2|^3} - gm_2 \left[\frac{r_1 - r_2}{|r_1 - r_2|^3} + \frac{r_2}{|r_2|^3}\right]$$

$$\ddot{r}_2 = -g(m + m_2) \frac{r_2}{|r_2|^3} - gm_1 \left[\frac{r_2 - r_1}{|r_2 - r_1|^3} + \frac{r_1}{|r_2|^3}\right]$$

where $|r_1|$ has been replaced by $|r_2|$

$$G = r_1^2 - r_2^2$$

$$\dot G = 2r_1 \cdot \dot r_1 - 2r_2 \cdot \dot r_2$$

$$\ddot G = 2\dot r_1^2 - 2\dot r_2^2 - 2g\left[\frac{m+m_1}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right] G$$
$$- 2g(m_1 - m_2)\left[\frac{1}{|r_2|^3} - \frac{1}{|r_1-r_2|^3}\right](r_2^2 - r_1 \cdot r_2)$$

$$\dddot G = -2gG\frac{d}{dt}\left[\frac{m+m_1}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right] - 4g\left[\frac{m_1+m}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right]\dot G$$
$$- 2g(m_1 - m_2)\left\{ 2r_2 \cdot \dot r_2\left(\frac{1}{|r_2|^3} - \frac{1}{|r_1-r_2|^3}\right) + \frac{d}{dt}\left[\left(\frac{1}{|r_2|^3} - \frac{1}{|r_1-r_2|^3}\right)(r_2^2 - r_1 \cdot r_2)\right]\right\}$$
$$- 4g(m_2 r_2 \cdot \dot r_1 - m_1 \dot r_2 r_1)\left(\frac{1}{|r_2|^3} - \frac{1}{|r_1-r_2|^3}\right)$$

Let $H = r_2 \cdot \dot r_1 - \dot r_2 \cdot r_1$ and $K = (r_1 - r_2)^2 - r_2^2$. If either $H = 0$ were an integral, and $m_1 = m_2$ or $K = 0$ were an integral the differential equation for G would reduce to

$$\dddot G = -2gG\frac{d}{dt}\left[\frac{m+m_1}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right] - 4g\left[\frac{m+m_1}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right]\dot G$$

This suggests that we should (a) set $m_1 = m_2$ and see if the constraints can be chosen so that G and H are simultaneous normalized pseudo-integrals, or (b) see if the constraints can be chosen so that G and K are simultaneous normalized pseudo-integrals.

Case (a). For this case

$$\dddot G = -2gG\frac{d}{dt}\left[\frac{m+m_2}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right]$$
$$- 4g\left[\frac{m+m_2}{|r_2|^3} + \frac{m_2}{|r_1-r_2|^3}\right]\dot G - 4gm_2H\left(\frac{1}{|r_2|^3} - \frac{1}{|r_1-r_2|^3}\right)$$

$$\dot H = gm_2G\left(\frac{1}{|r_2|^3} - \frac{1}{|r_1-r_2|^3}\right)$$

Thus H and G will be simultaneous normalized pseudo integral if at $t = \hat{t}$ (and consequently see above, for all t in a domain)

$$H = G = \dot{G} = \ddot{G} = 0$$

that is, at $t = \hat{t}$,

$$r_2^2 = r_1^2$$

$$\dot{r}_2^2 = \dot{r}_1^2$$

$$r_2 \cdot \dot{r}_2 = r_1 \cdot \dot{r}_1$$
$$r_2 \cdot \dot{r}_1 = r_1 \cdot \dot{r}_2$$

If we let (p_i, θ_i) and (v_i, α_i) be the polar coordinates of r_i and $\dot{r}_i$ at $t = \hat{t}$ and make use of trigonometric identities for

$$\cos(\theta_2 - \alpha_1) = \cos[(\theta_2 - \alpha_2) + (\alpha_2 - \alpha_1)]$$

and

$$\cos(\theta_1 - \alpha_2) = \cos[(\theta_1 - \alpha_1) - (\alpha_2 - \alpha_1)]$$

these constraints are equivalent to (we assume p_1 and p_2 are not 0 since this is a singularity of the system of differential equations)

(i) $p_2 = p_1$
(ii) $v_2 = v_1$
(iii) $v_1 = 0$ or $\alpha_2 - \alpha_1 = n\pi$ and $\theta_2 - \alpha_2 = \pm(\theta_1 - \alpha_1) + 2l\pi$ or $\theta_2 - \alpha_2 = -(\theta_1 - \alpha_1) + 2l\pi$ where n and l are integers.

Case (b). If we replace both $| r_1 - r_2 |$ and $| r_1 |$ by $| r_2 |$ in the differential equations of motion these equations become

$$\ddot{r}_1 = Lr_1$$
$$\ddot{r}_2 = Lr_2$$

where $L = -g[(m + m_1 + m_2)/| r_2 |^3]$. As above if G and K are simultaneous normalized pseudo-integrals for this system they are also simultaneous normalized pseudo-integrals for the original system. It readily follows that

$$G = r_1^2 - r_2^2$$

$$K = (r_1 - r_2)^2 - r_2^2 = r_1^2 - 2r_1 \cdot r_2$$

$$\dot{G} = 2r_1 \cdot \dot{r}_1 - 2r_2 \cdot \dot{r}_2$$

$$\dot{K} = 2r_1 \cdot \dot{r}_1 - 2\dot{r}_1 \cdot r_2 - 2r_1 \cdot \dot{r}_2$$

$$\ddot{G} = 2\dot{r}_1^2 - 2\dot{r}_2^2 + 2LG$$

$$\ddot{K} = 2\dot{r}_1^2 - 4\dot{r}_1 \cdot \dot{r}_2 + 2LK$$

$$\dddot{G} = 4L\dot{G} + 2\dot{L}G$$

$$\dddot{K} = 4L\dot{K} + 2\dot{L}K$$

Thus G and K are simultaneous normalized pseudo-integrals if at $t = \hat{t}$

$$r_2^2 = r_1^2$$

$$\dot{r}_2^2 = \dot{r}_1^2$$

$$r_2 \cdot \dot{r}_2 = r_1 \cdot \dot{r}_1$$

$$r_1 \cdot r_2 = \tfrac{1}{2} r_1^2$$

$$\dot{r}_1 \cdot \dot{r}_2 = \tfrac{1}{2} \dot{r}_1^2$$

$$r_1 \cdot \dot{r}_1 = \dot{r}_1 \cdot r_2 + r_1 \cdot \dot{r}_2$$

Using the same notation as for case (a) we find that these conditions are equivalent to (again p_1, and p_2 are not 0)

(i) $p_2 = p_1$
(ii) $v_2 = v_1$
(iii) $\vartheta_2 - \theta_1 = \pm \pi/3 + 2l\pi$
(iv) $v_1 = 0$ or $\theta_2 - \alpha_2 = \theta_1 - \alpha_1 + 2n\pi$, where n and l are integers.

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